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Infinitesimally holomorphic curves in hyperkähler four-manifolds

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Dedicated to Professor Koichi Ogiue on his sixtieth birthday

Abstract

We prove that closed infinitesimally holomorphic curves in a hyperkähler four-manifold are actually holomorphic with respect to one of the parallel complex structures on the ambient space compatible with the metric.

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1. Statements of the results

In our previous paper [3], we posed the following problem.

Problem. Let Q^{2n+m} be a $(2n+m)$-dimensional Riemannian spin manifold with a nonzero parallel spinor field. Let M^{2n} be a $2n$-dimensional closed Riemannian spin manifold isometrically immersed in Q^{2n+m}. Then classify all submanifolds M^{2n} satisfying the condition that there exists a nonzero parallel spinor field $\psi \in \Gamma(\Sigma Q)$ such that

$$\nabla^{\Sigma M \otimes \Sigma N}(\psi|_M) = 0. \qquad (*)$$

Here $\psi|_M$ denotes the restriction of ψ to the submanifold M^{2n}.

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A four-dimensional Riemannian spin manifold Q^4 with a nonzero parallel spinor field is nothing but a hyperkähler four-manifold. If Q^4 is simply connected, then a Ricci-flat Kähler structure on Q^4 coincides with a hyperkähler structure. K3 surfaces equipped with Ricci-flat Kähler metrics and four-dimensional hyperkähler ALE spaces are typical examples of such Q^4.

For the above problem, in the paper [3], we gave the following answer in the case where $n = 1$ and $m = 2$.

Theorem 1.1 ([3, Theorem 4.5]). *Let Q^4 be a hyperkähler four-manifold. Let M^2 be a two-dimensional closed Riemannian spin manifold isometrically immersed in Q^4. Then the following conditions are equivalent:*

(i) *M^2 is a holomorphic curve with respect to one of the complex structures on Q^4 compatible with the metric.*
(ii) *M^2 satisfies the condition $(*)$.*

On the other hand, we can consider the condition $(*)$ as a purely local condition without the assumption that M^{2n} is *closed* (i.e. compact and without boundary). In this paper, we characterize the condition $(*)$ by the second fundamental form in the case where $n = 1$ and $m = 2$.

Proposition 1.2. *Let Q^4 be a four-dimensional Riemannian spin manifold with a nonzero parallel spinor field. Let M^2 be a two-dimensional Riemannian spin manifold isometrically immersed in Q^4. Then M^2 satisfies the condition $(*)$ if and only if the components of the second fundamental form of M^2 in Q^4 satisfy that*

$$h^3_{11} = h^4_{12} = -h^3_{22}, \qquad h^4_{22} = h^3_{12} = -h^4_{11}.$$

The above equations are closely related to the notion of infinitesimally holomorphic immersions. We recall its definition (cf. [5, Section 3]).

Definition. An immersion $F : M^2 \to \tilde{M}^4$ of an oriented surface into an oriented Riemannian four-manifold is said to be *infinitesimally holomorphic* if there exists a parallel complex structure J on $F^*(T\tilde{M}^4)$ such that

$$F_* \circ J_M = J \circ F_*,$$

where J_M is the complex structure on M^2.

Micallef and Wolfson characterized the infinitesimally holomorphic immersion in terms of the second fundamental form.

Lemma 1.3 (Micallef–Wolfson [5]). *An immersion $F : M^2 \to \tilde{M}^4$ of an oriented surface into an oriented Riemannian four-manifold is infinitesimally holomorphic if and only if*

$$h^3_{11} = h^4_{12} = -h^3_{22}, \qquad h^4_{22} = h^3_{12} = -h^4_{11} \tag{1}$$

or

$$h^3_{22} = h^4_{12} = -h^3_{11}, \qquad h^4_{11} = h^3_{12} = -h^4_{22}. \tag{2}$$

Therefore Proposition 1.2 implies that if M^2 satisfies the condition $(*)$, then M^2 is an infinitesimally holomorphic curve. The converse is also true under an appropriate choice of orientation of M^2.

Moreover, using Theorem 1.1, we shall prove the following theorem.

Theorem 1.4. *Let Q^4 be a hyperkähler four-manifold. Let M^2 be an oriented closed surface immersed in Q^4. Let M^2 carry the induced Riemannian metric. If M^2 is infinitesimally holomorphic in Q^4, then M^2 is a holomorphic curve with respect to one of the complex structures on Q^4 compatible with the metric.*

A holomorphic curve is of course infinitesimally holomorphic. Let T^4 be a four-dimensional torus with its standard flat metric. It is known that an infinitesimally holomorphic curve in T^4 is holomorphic (cf. [5, Section 3]). Theorem 1.4 is a generalization of this result to the case of closed surfaces in hyperkähler four-manifolds.

2. Proof of Proposition 1.2

We shall employ the notation of our previous paper [3]. Let Q^4 be a four-dimensional Riemannian spin manifold with a nonzero parallel spinor field ψ. Without loss of generality, we may assume that ψ is a positive spinor field and its norm is identically equal to 1. Let M^2 be a two-dimensional Riemannian spin manifold isometrically immersed in Q^4.

ΣQ, ΣM and ΣN denote the complex spinor bundles associated to spin structures of the tangent bundle TQ of Q^4, the tangent bundle TM of M^2 and the normal bundle N of M^2 in Q^4, respectively.

Let $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ be oriented local orthonormal frames of TM and N, respectively. Then the components of the second fundamental form II are defined by

$$h_{ij}^k := \langle II(X_i, X_j), Y_{k-2}\rangle,$$

where $i, j = 1, 2$ and $k = 3, 4$. Remark that $h_{ij}^k = h_{ji}^k$.

The restriction $\psi|_M$ of ψ to the surface M^2 belongs to the sections of $\Sigma Q|_M (= \Sigma M \otimes \Sigma N)$ and $\langle \psi|_M, \psi|_M\rangle \equiv 1$. Then we have the following equation

$$\nabla_X^{\Sigma Q}(\psi|_M) - \nabla_X^{\Sigma M \otimes \Sigma N}(\psi|_M) = \frac{1}{2}\sum_{i=1}^{2} \gamma_Q(X_i \cdot II(X, X_i))\psi|_M, \tag{3}$$

where γ_Q means the Clifford multiplication on $\Sigma Q|_M$ (see [1, Section 2] or [3, Section 2]). By (3), the condition $(*)$ is equivalent to the following equations:

$$\gamma_Q(X_1 \cdot II(X_1, X_1))\psi|_M + \gamma_Q(X_2 \cdot II(X_1, X_2))\psi|_M = 0, \tag{4}$$

$$\gamma_Q(X_1 \cdot II(X_2, X_1))\psi|_M + \gamma_Q(X_2 \cdot II(X_2, X_2))\psi|_M = 0. \tag{5}$$

Here we show the following identities.

Lemma 2.1. *For any positive parallel spinor field* ψ *on* Q^4 *such that* $|\psi| \equiv 1$, *we have*

$$|\gamma_Q(X_1 \cdot II(X_1, X_1))\psi|_M + \gamma_Q(X_2 \cdot II(X_1, X_2))\psi|_M|^2 \\ = (h^3_{11} - h^4_{12})^2 + (h^4_{11} + h^3_{12})^2, \tag{6}$$

$$|\gamma_Q(X_1 \cdot II(X_2, X_1))\psi|_M + \gamma_Q(X_2 \cdot II(X_2, X_2))\psi|_M|^2 \\ = (h^3_{21} - h^4_{22})^2 + (h^4_{21} + h^3_{22})^2, \tag{7}$$

where $|\cdot|$ *denotes the point-wise norm defined by the Hermitian metric* $\langle\cdot,\cdot\rangle$ *on the complex vector bundle* $\Sigma Q|_M$ *on* M^2.

Proof. We first prove (6).

$$\begin{aligned}
&|\gamma_Q(X_1 \cdot II(X_1, X_1))\psi|_M + \gamma_Q(X_2 \cdot II(X_1, X_2))\psi|_M|^2 \\
&\quad = |II(X_1, X_1)|^2\langle\psi|_M, \psi|_M\rangle + |II(X_1, X_2)|^2\langle\psi|_M, \psi|_M\rangle \\
&\qquad + 2\mathrm{Re}\langle\gamma_Q(X_1 \cdot II(X_1, X_1))\psi|_M, \gamma_Q(X_2 \cdot II(X_1, X_2))\psi|_M\rangle \\
&\quad = |II(X_1, X_1)|^2 + |II(X_1, X_2)|^2 \\
&\qquad + 2\mathrm{Re}\langle\gamma_Q(X_1 \cdot (h^3_{11}Y_1 + h^4_{11}Y_2))\psi|_M, \gamma_Q(X_2 \cdot (h^3_{12}Y_1 + h^4_{12}Y_2))\psi|_M\rangle \\
&\quad = |II(X_1, X_1)|^2 + |II(X_1, X_2)|^2 + 2h^3_{11}h^3_{12}\mathrm{Re}\langle\gamma_Q(X_1 \cdot Y_1)\psi|_M, \gamma_Q(X_2 \cdot Y_1)\psi|_M\rangle \\
&\qquad + 2h^3_{11}h^4_{12}\mathrm{Re}\langle\gamma_Q(X_1 \cdot Y_1)\psi|_M, \gamma_Q(X_2 \cdot Y_2)\psi|_M\rangle \\
&\qquad + 2h^4_{11}h^3_{12}\mathrm{Re}\langle\gamma_Q(X_1 \cdot Y_2)\psi|_M, \gamma_Q(X_2 \cdot Y_1)\psi|_M\rangle \\
&\qquad + 2h^4_{11}h^4_{12}\mathrm{Re}\langle\gamma_Q(X_1 \cdot Y_2)\psi|_M, \gamma_Q(X_2 \cdot Y_2)\psi|_M\rangle \\
&\quad = |h^3_{11}Y_1 + h^4_{11}Y_2|^2 + |h^3_{12}Y_1 + h^4_{12}Y_2|^2 + 2h^3_{11}h^3_{12}\mathrm{Re}\langle\gamma_Q(X_1)\psi|_M, \gamma_Q(X_2)\psi|_M\rangle \\
&\qquad + 2h^3_{11}h^4_{12}\mathrm{Re}\langle\gamma_Q(X_1 \cdot X_2 \cdot Y_1 \cdot Y_2)\psi|_M, \psi|_M\rangle \\
&\qquad + 2h^4_{11}h^3_{12}\mathrm{Re}\langle\gamma_Q(-X_1 \cdot X_2 \cdot Y_1 \cdot Y_2)\psi|_M, \psi|_M\rangle \\
&\qquad + 2h^4_{11}h^4_{12}\mathrm{Re}\langle\gamma_Q(X_1)\psi|_M, \gamma_Q(X_2)\psi|_M\rangle \\
&\quad = (h^3_{11})^2 + (h^4_{11})^2 + (h^3_{12})^2 + (h^4_{12})^2 - 2h^3_{11}h^4_{12} + 2h^4_{11}h^3_{12} \\
&\quad = (h^3_{11} - h^4_{12})^2 + (h^4_{11} + h^3_{12})^2.
\end{aligned}$$

Here we used the facts that $\mathrm{Re}\langle\gamma_Q(X_i)\psi|_M, \gamma_Q(X_j)\psi|_M\rangle = 0$ for $i \neq j$ and that $\omega_{\mathbf{C}} := -X_1 \cdot X_2 \cdot Y_1 \cdot Y_2$ is the chirality operator on $(\Sigma Q)_p$ for each $p \in M^2$. Eq. (7) is also proved by the same calculation. □

If M^2 satisfies the condition $(*)$, then Lemma 2.1 yields that

$$h^3_{11} - h^4_{12} = 0, \qquad h^4_{11} + h^3_{12} = 0, \qquad h^3_{21} - h^4_{22} = 0, \qquad h^4_{21} + h^3_{22} = 0.$$

Hence we have the condition (1).

Conversely, if the second fundamental form of M^2 satisfies (1), then we have Eqs. (4) and (5) by Lemma 2.1. Therefore, we obtain the condition $(*)$.

Thus we complete the proof of Proposition 1.2.

3. Proof of Theorem 1.4

Let Q^4 be a hyperkähler four-manifold. Indeed, Q^4 is a spin manifold and has a nonzero positive parallel spinor field ψ (see [2, Chapter 6]). Let M^2 be an oriented closed surface immersed in Q^4. Let M^2 carry the induced Riemannian metric. Since the second Stiefel–Whitney class of M^2 is zero, there exists a spin structure on M^2. We fix a spin structure on M^2. Then the normal bundle N carries the induced spin structure (see [4, p. 85]). Suppose that M^2 is infinitesimally holomorphic in Q^4. By Lemma 1.3, the components of the second fundamental form II of M^2 in Q^4 satisfy the condition (1) or (2).

Case 1: II satisfies (1).

By Proposition 1.2, M^2 satisfies the condition $(*)$. By Theorem 1.1, M^2 is a holomorphic curve with respect to one of the complex structures on Q^4 compatible with the metric.

Case 2: II satisfies (2).

$\bar{M}^2$ denote the surface M^2 with the opposite orientation. When the second fundamental form of M^2 satisfies (2), that of $\bar{M}^2$ satisfies (1). By Proposition 1.2, $\bar{M}^2$ satisfies the condition $(*)$. By Theorem 1.1, $\bar{M}^2$ is a holomorphic curve with respect to one of the complex structures on Q^4 compatible with the metric. We denote its complex structure by J. M^2 is anti-holomorphic with respect to J. In other words, M^2 is holomorphic with respect to $-J$.

Thus we finish the proof of Theorem 1.4.

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